Title	JC and Polytechnic Mathematics Materials Compilation – Pure Mathematics
Author	AprilDolphin
Date	22/12/2024

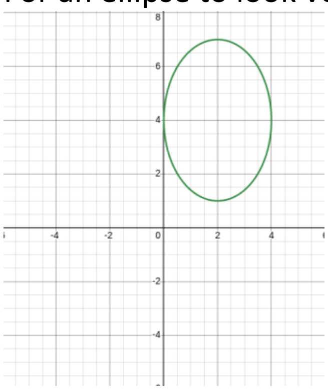
Title	Graphs and Functions – Conic Sections – Hyperbola and Ellipse
Author	AprilDolphin
Date	21/12/2024
Note	This article assume you already understand the properties of circle and parabola as taught in secondary school Additional Mathematics and Elementary Mathematics respectively.

In general, all conic sections can be represented using the equation below,
$Ax^2 + Bx + Cy^2 + Dy + E = 0$
Different types of conic sections appears when we begin changing the value of A, B, C, D and E

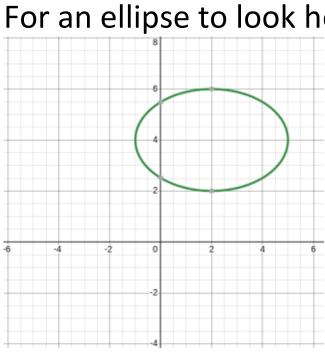
If the conic section is an Ellipse, it has the following equation required in general: (Make sure the right-hand side is exactly 1 before proceeding!)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

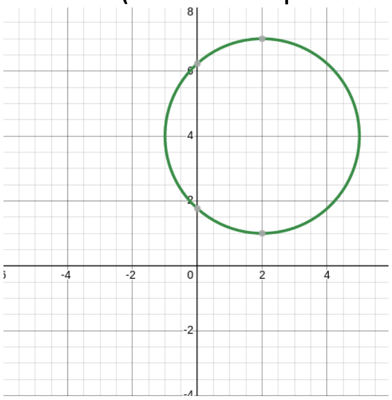
For an ellipse to look vertically stretched, $a < b$



For an ellipse to look horizontally stretched, $a > b$



For an ellipse to have an exactly circular shape, $a = b$, where a represents the radius of a circle. (Circle is a special case of an ellipse.)



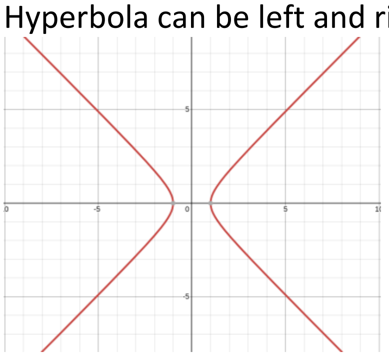
Properties of an ellipse.
The point where the centre lies is $C(h, k)$
The coordinates of vertices from up, down, left and right can be represented as the following

Upper Vertex (Top of Ellipse)	$U(h, k + b)$
Lower Vertex (Bottom of Ellipse)	$D(h, k - b)$
Left Vertex (Left side of Ellipse)	$L(h - a, k)$
Right Vertex (Right side of Ellipse)	$R(h + a, k)$

The lines of symmetry of an ellipse are represented by $x = h, y = k$

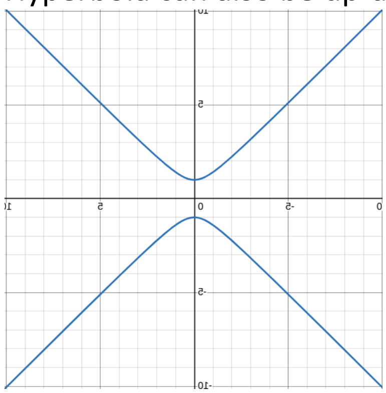
If the conic section is a Hyperbola, it has the following equation in general: (Make sure the right-hand side is exactly 1 before proceeding!)

Hyperbola can be left and right opening as seen with the below diagram



Properties of left-and-right-opening hyperbola as follows:
Can be reduced to the following equation, $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where RHS is exactly 1

Hyperbola can also be up-and-down-opening as seen with the below diagram



Properties of up-and-down-opening hyperbola as follows:

Can be reduced to the following equation, $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$, where RHS is exactly 1

Further Properties of Hyperbola		
	Left-and-Right-Opening	Up-and-Down-Opening
Centre	$C(h,k)$	$C(h,k)$
Vertices	Left Vertex, $L(h-a,k)$ Right Vertex, $R(h+a,k)$	Upper Vertex, $U(h,k+b)$ Bottom Vertex,$B(h,k-b)$
Line of symmetry	$x=h$ $y=k$	$x=h$ $y=k$
Asymptotes	$y=k+\frac{b}{a}(x-h)$ $y=k-\frac{b}{a}(x-h)$	$y=k+\frac{b}{a}(x-h)$ $y=k-\frac{b}{a}(x-h)$

Title	JC and Polytechnic Mathematics – Basic Operation of Complex Numbers
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	8/12/2018

This portion is designed to address a basics of complex numbers addition, subtraction, multiplication and division and does not involve any geometry.

Applicable to
- JC 'A' Level H2 Mathematics
- Nanyang Polytechnic, School of Engineering – Engineering Mathematics 1A
- Nanyang Polytechnic, School of Information Technology – Diploma Plus Program (Mathematics)

Introducing the imaginary unit i (Engineering students use j instead)
$i = \sqrt{-1}$

If you have read my document on secondary school algebra, you will realize that I specifically tell students not to use i in any algebra operations (unless question ask you to do so, of course) as it is reserved for specific use. For a long time, you may be thinking what exactly it is being used for, it is in fact used to represent the value of $\sqrt{-1}$ at higher level mathematics.

Basic Operation of Imaginary Numbers
Addition Rule
$xi + yi = (x + y)i$

Subtraction Rule
$xi - yi = (x - y)i$

Multiplication Rule
$i(i) = -1$
$(-i)(i) = 1$
$xi\,(yi) = x(y)(-1) = -xy$

Division Rule
$$\frac{xi}{yi} = \frac{x}{y}$$

Basic Examples:
Q1. $6i + 8i$
Q2. $5i - 3i$
Q3. $3i(4i)$
Q4. $\frac{2i}{5i}$

Q1	$6i + 8i = 14i$
Q2	$5i - 3i = 2i$
Q3	$3i(4i) = -12$
Q4	$\frac{2i}{5i} = \frac{2}{5}$

If a number contains both an imaginary and a real number, the number is called a complex number.

A complex number is in the following form

$$a + bi$$

a is the real part, bi is the imaginary part.

Complex Number Operations
Addition Rule
$(a + bi) + (g + hi) = (a + g) + (b + h)i$

Subtraction Rule
$(a + bi) - (g + hi) = (a - g) + (b - h)i$

Multiplication Rule (I hope you recall your secondary school expansion and factorization class, as it is similar to that, with one exception, that $i(i) = -1$)
$(a + bi)(g + hi) = ag + bgi + ahi + (-1)bh$
$ag - bh + bgi + ahi = (ag - bh) + (bg + ah)i$

Division Rule (Students who took Additional Mathematics, this may ring a bell as it is similar to the method you used to rationalize denominator in secondary 3.)

Situation 1.
$$\frac{a + bi}{g + hi} = \frac{a + bi}{g + hi} \times \frac{g - hi}{g - hi}$$

Situation 2.

$$\frac{a+bi}{g-hi} = \frac{a+bi}{g-hi} \times \frac{g+hi}{g+hi}$$

This method is called complex conjugate multiplication, a complex conjugate has the following properties that makes the above method work.

1. The conjugate multiplied to the complex number will always be equal to 1.
2. The sign between the imaginary and real part of the denominator has to flip to become the conjugate. (i.e. "+" to "-" and "-" to "+")

Basic Examples

Q5. $(3-6i)-(7-8i)$

Q6. $(15+6i)+(5-3i)$

Q7. $(5+i)(6-3i)$

Q8. $\frac{2+5i}{1+6i}$

Q5.	By distributive law, the expression is equal to the following $3-6i-7+8i=$ $3-7-6i+8i=$ $-4+2i$
Q6.	$(15+6i)+(5-3i)=$ $20+3i$
Q7.	(Like how you expand expression brackets in secondary school, with an exception to note, being $i \times i = -1$) $(5+i)(6-3i)$ $5(6)+6(i)-15i-3(i)(i)=$ $30-9i-3(-1)=$ $33-9i$

Q8.	The complex conjugate of the denominator of the expression $\frac{2+5i}{1+6i}$ is $1-6i$ $\frac{2+5i}{1+6i}\times\frac{1-6i}{1-6i}$ $=\frac{(2+5i)(1-6i)}{(1+6i)(1-6i)}$ $=\frac{2+5i-12i+30}{1-36i(i)}$ $=\frac{32-7i}{1+36}$ $=\frac{32-7i}{37}$

Title	Complex Numbers – Argument and Modulus [Converting from Algebraic Form to Polar and Exponential Form Form]
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	15/12/2018

Applicable to

- Junior College 'A' Level H2 Mathematics
- Nanyang Polytechnic, School of Engineering, Engineering Mathematics 1A
- Nanyang Polytechnic, School of Information Technology, Mathematics (Diploma Plus Program)

Basic Notation	
r	The modulus of the complex number. (Also denoted in textbooks using the 2 vertical strokes. If z is the complex number, the modulus can be written as $\lvert z\rvert$.
$\theta_{argument}$	The argument of the complex numbers. (Denoted in textbooks as $\arg(z)$, given z is the complex number.)

Notation Example	
$a + bi$ (Algebraic Form)	Algebraic Interpretation: a is the real part of the complex number and b is the imaginary part of the complex number.
$r\angle\theta$ (Polar Coordinate Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus) θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)
$r(\cos(\theta) + i\sin(\theta))$ (Polar Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus)

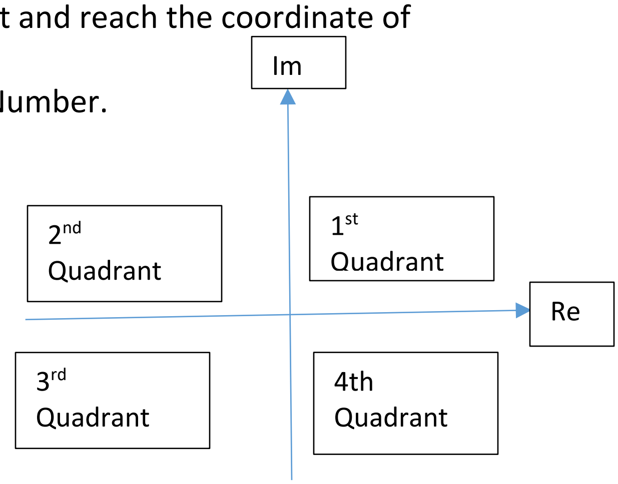
	θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)
$re^{i\theta}$ (Exponential Form)	Given complex number $a + bi$ r refers to the distance of the point on the complex plane relative to the origin point of the complex plane. Can be computed using $r = \sqrt{a^2 + b^2}$ (Also known as the modulus) θ refers to angle of the line relative the positive real-axis, that start from the point of origin to the coordinate of $a + bi$ (Also known as the argument)

As you can see the required values for polar coordinate, polar form and exponential form are computed using the same formula, so I will just need to guide you all on how to compute the modulus and argument.

<u>(Make sure your calculator is in “Radian Mode” before proceeding)</u>

Step 1. Sketch a complex plane and plot the coordinate of the complex number within the complex plane. (This is also called an argand diagram.)
Step 2. Draw a line, the line shall start from the origin point and reach the coordinate of the complex number.
Step 3. Determine the Quadrant Number of the Complex Number.

(Im means imaginary axis, Re means the real axis.)



Step 4. Compute Basic Angle of the Complex Number
Using $\theta_{basic} = \tan^{-1}\left(\frac{Opposite}{Adjacent}\right)$

Step 5. Use the following rules to deduce the argument
For first quadrant
$\theta_{argument} = \theta_{basic}$

For second quadrant
$\theta_{argument} = \pi - \theta_{basic}$

For third quadrant
$\theta_{arguement} = -\pi + \theta_{basic}$

For fourth quadrant
$\theta_{argument} = -\theta_{basic}$

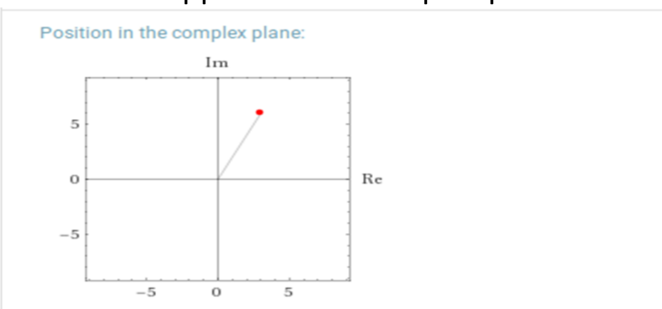
Step 6. Compute r using the below formula
$r = \sqrt{a^2 + b^2}$

Find the modulus and argument of the following complex numbers.

1. $3 + 6i$
2. $1 - 3i$
3. $-1 + 3i$
4. $-1 - i$

Question 1.
First quadrant therefore:
$\theta_{argument} = \theta_{basic}$

$3 + 6i$ as mapped on the complex plane

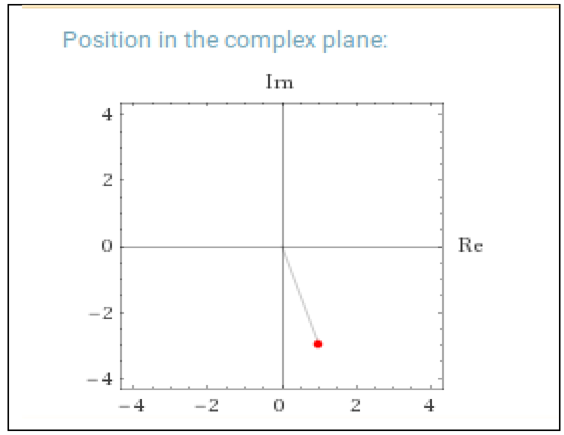


$$\theta = \tan^{-1}\left(\frac{6}{3}\right) = 1.11\ Radians$$

$$r = \sqrt{6^2 + 3^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$$

Written in Polar Coordinate Form: $3\sqrt{5} \angle 1.11$
Written in Polar Form $3\sqrt{5}\ (\cos(1.11) + i\sin(1.11))$
Written in Exponential Form $3\sqrt{5}e^{i\,(1.11)}$

Question 2.
$1 - 3i$ as mapped on the complex plane.



The complex position of the value lies in the fourth quadrant of the complex plane, therefore, $\theta_{argument} = -\theta_{basic}$
In this case, the opposite is the absolute value of the imaginary part while the adjacent is the real part of the complex number.

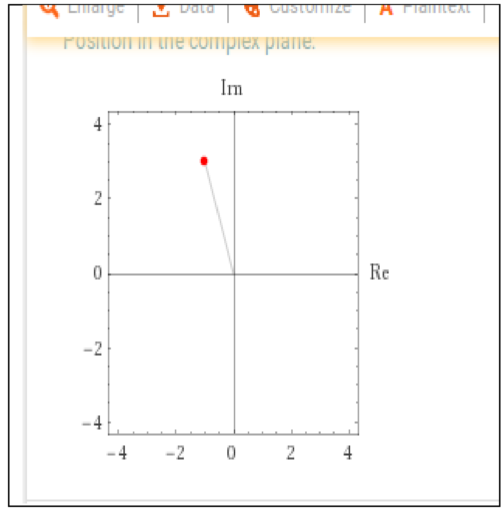
*When dealing with angles, always convert coordinate values to absolute values.

$$\theta_{basic} = \tan^{-1}\left(\frac{3}{1}\right) = 1.249\ Radians$$
$$\theta_{argument} = -1.249\ Radians$$
$$r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

Written in Polar Coordinate Form:	$\sqrt{10}\angle - 1.25 Radians$
Written in Polar Form	$\sqrt{10}(\cos(-1.25) + i\sin(-1.25))$
Written in Exponential Form	$\sqrt{10}\, e^{-1.25\, i}$

Question 3.

$-1 + 3i$ as mapped on the complex plane



Since in Second Quadrant, the Argument is computed as follows

$$\theta_{argument} = \pi - \theta_{basic}$$

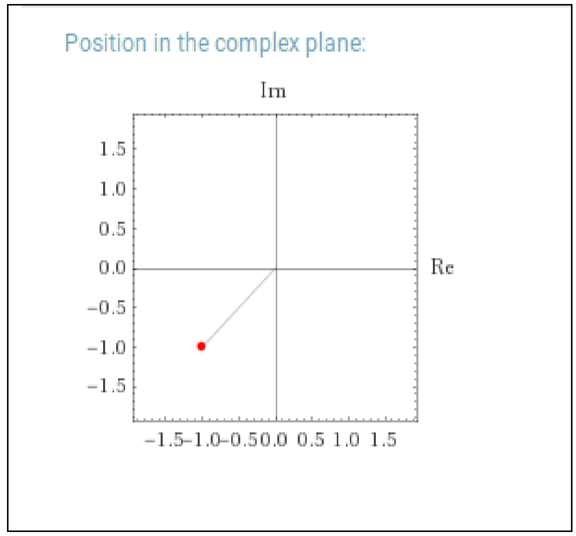
$$\theta_{basic} = \tan^{-1}\left(\frac{3}{1}\right) = 1.24905\ Radians$$

$$\theta_{argument} = \pi - 1.24905\ Radians = 1.89\ Radians$$

$$r = \sqrt{3^2 + 1^2} = \sqrt{10}$$

Written in Polar Coordinate Form	$\sqrt{10}\ \angle 1.89\ Radians$
Written in Polar Form	$\sqrt{10}\ (\cos(1.89) + i\sin(1.89))$
Written in Exponential Form	$\sqrt{10}(\mathrm{e}^{1.89i})$

Question 4.
$-1 - i$ as mapped on the complex plane.



Since the coordinate of the complex number lies in the third quadrant, we use the following formula

$$\theta_{arguement} = -\pi + \theta_{basic}$$

$$\theta_{basic} = \tan^{-1}\left(\frac{1}{1}\right) = 0.78540\ Radians$$

$$\theta_{arguement} = -\pi + 0.78540 = -2.36\ Radians$$

$$r = \sqrt{1^2 + 1^2} = 1.4142$$

Written in Polar Coordinate Form	$\sqrt{2}\angle - 2.36\ Radians$
Written in Polar Form	$\sqrt{2}\,(\sin(-2.36) + i\sin(-2.36))$
Written in Exponential Form	$\sqrt{2}\,(e^{-2.36i})$

Title	Differentiation of Inverse Trigonometric Functions
Author	AprilDolphin
Date	19/9/2024
Assumptions	This article assumes you already have good knowledge of using chain rule to differentiate functions as well as basic knowledge of finding derivatives of algebraic, logarithmic, exponential and trigonometric functions.

Function	Derivatives
$f(x) = \sin^{-1}(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = \cos^{-1}(x)$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}$
$f(x) = \tan^{-1}(x)$	$f'(x) = \frac{1}{1+x^2}$

Chain Rule in General
Given $f(x) = g\big(h(x)\big)$ $f'(x) = g'(x)\, h'(x)$

Application of Chain Rule to Inverse Trigonometric Functions	
Function	Derivatives
$f(x) = \sin^{-1}(u)$	$f'(x) = \frac{1}{\sqrt{1-u^2}}(u')$
$f(x) = \cos^{-1}(u)$	$f'(x) = \frac{-1}{\sqrt{1-u^2}}(u')$
$f(x) = \tan^{-1}(u)$	$f'(x) = \frac{1}{1+u^2}(u')$

Example 1.
Find the derivative of the following functions

(a) $\mathrm{y} = \sin^{-1}(e^{6t})$

Using chain rule $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$

$$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}} \times 6e^{6t} = \frac{6e^{6t}}{\sqrt{1-e^{12t}}}$$

(b) $y = \tan^{-1}(2^x)$

Using chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{1}{1+(2^x)^2} \times 2^x \ln(2) = \frac{2^x \ln(2)}{1+2^{2x}}$$

(c) $f(x) = \cos^{-1}\left(\frac{x}{3}\right)$

Using chain rule, where any composite function $f(x) = g\big(h(x)\big)$ will produce a derivative of $g'(x)h'(x)$.

$$\cos^{-1}\left(\frac{x}{3}\right) = \cos^{-1}\left(\frac{1}{3}x\right)$$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{3}x\right)^2}} \times \frac{1}{3} = -\frac{1}{3\sqrt{1-\frac{1}{9}x^2}} = -\frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}$$

$$= -\frac{1}{\sqrt{9-\frac{\sqrt{9^2}x^2}{9}}} = -\frac{1}{\sqrt{9-x^2}}$$

(d) $y = e^{\sin^{-1}(2x)}$

Using chain rule, where $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$2e^{\sin^{-1}(2x)} \times \frac{1}{\sqrt{1-(2x^2)}} = \frac{2e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}$$

Title	Differentiation of Implicit Functions
Author	AprilDolphin
Date	27/9/2024

Definition of an explicit function, is where
y can be defined as exactly equal to $f(x)$ or written mathematically, $y = f(x)$

Definition of an implicit function is where
y cannot be written as equal to solely just $f(x)$ but requiring the person to write or rewrite a function as where $g(x, y) = 0$

Examples of Explicit Functions	Examples of Implicit Functions
$y = x^2 + \ln(x) + 2x + 4$	$x^2 + y^2 + 2xy = 25$
$y = \sin(x^4) + x^5 - \tan(x)$	$y^3 - \frac{5x}{y} = 6x^2y^6 + 9x - 64$
$y = e^{\sin(x)} - \cos(x) + 5$	$y^5 + \sin(xy) = x^2 + y^5$

Rules of Differentiation Necessary to Ensure a Smooth Understanding of Implicit Differentiation:

Chain Rule: $f(x) = g\big(h(x)\big), f'(x) = g'(x)\ h'(x)$

Product Rule: $f(x) = g(x)\ h(x), f'(x) = g'(x)\ h(x) + h'(x)\ g(x)$

Quotient Rule: $f(x) = \frac{g(x)}{h(x)}, f'(x) = \frac{h(x)g'(x) - g(x)h\prime(x)}{[h(x)]^2}$

Instruction to Implicit Differentiation

1. Differentiate left side and right side of implicit function with respect to x
2. Identify term where y is differentiated, multiplying the derivative of $f(y)$ by $\frac{dy}{dx}$
3. Group terms with $\frac{dy}{dx}$ together on one side and without on the other side
4. Take out common factor of $\frac{dy}{dx}$ on the side with the term of $\frac{dy}{dx}$
5. Remove terms without the factor of $\frac{dy}{dx}$ by division, multiplication or any applicable method

Example 1.
Find the derivative of the following implicit function
$y^2x^2 + x = y - 10$

Step 1. Differentiate left hand side and right-hand side of implicit function with respect to x.

$$\frac{d}{dx}\,[y^2x^2 + x] = \frac{d}{dx}[y - 10]$$

Step 2. Differentiating term by term, we get the following and can identify the term where y is differentiated then multiply the derivative by $\frac{dy}{dx}$

Terms	Derivatives
y^2x^2 using product rule →	$[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2)$
x using power rule →	1
y	$1\times \frac{dy}{dx}$
-10	0

Putting all derivative together, we would get the following value:
$$[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2) + 1 = 1 \times \frac{dy}{dx}$$

Step 3. Group terms with $\frac{dy}{dx}$ together on one side and without on the other side

$$2yx^2\left(\frac{dy}{dx}\right) + 2xy^2 + 1 = \frac{dy}{dx}$$

$$2yx^2\left(\frac{dy}{dx}\right) - \frac{dy}{dx} = -2xy^2 - 1$$

Step 4. Take out common factor of $\frac{dy}{dx}$ on the side with the term of $\frac{dy}{dx}$

$$(2yx^2 - 1)\left(\frac{dy}{dx}\right) = -2xy^2 - 1$$

Step 5. Remove terms without the factor of $\frac{dy}{dx}$ by division, multiplication or any available method

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2yx^2 - 1}$$

Example 2. Differentiate the following implicit function

$y^2 = 2x^3 + y + 7$

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[2x^3 + y + 7]$$

$$2y\left(\frac{dy}{dx}\right) = 6x^2 + 1\left(\frac{dy}{dx}\right) + 0$$

$$2y\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 6x^2$$

$$\frac{dy}{dx}(2y - 1) = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{2y - 1}$$

Example 3.

$2xy + y^2 - 3x - x^2y^5 = 0$

$$\frac{d}{dx}[2xy + y^2 - 3x - x^2y^5] = \frac{d}{dx}[0]$$

Terms	Derivative
$2xy$	$2x(1) \times \frac{dy}{dx} + 2y$
y^2	$2y \times \frac{dy}{dx}$
$-3x$	-3
$-x^2y^5$	$-2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx}$

0	0

$$2x\left(\frac{dy}{dx}\right) + 2y + 2y\left(\frac{dy}{dx}\right) - 3 - 2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx} = 0$$

$$2x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + 2y - 3 - 2xy^5 - 5x^2y^4\left(\frac{dy}{dx}\right) = 0$$

$$2x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) - 5x^2y^4\left(\frac{dy}{dx}\right) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx}(2x + 2y - 5x^2y^4) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx} = \frac{-2y + 3 + 2xy^5}{2x + 2y - 5x^2y^4}$$

Example 4.

$$y + y^2 + xy = 5 - \sin(e^{3y})$$

$$\frac{d}{dx}[y + y^2 + xy] = \frac{d}{dx}[5 - \sin(e^{3y})]$$

$$1\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + y + x\left(\frac{dy}{dx}\right) = 0 - 3e^{3y}\cos e^{3y}\left(\frac{dy}{dx}\right)$$

$$1\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + x\left(\frac{dy}{dx}\right) + 3e^{3y}\cos(e^{3y})\left(\frac{dy}{dx}\right) = -y$$

$$\frac{dy}{dx}[1 + 2y + x + 3e^{3y}\cos(e^{3y})] = -y$$

$$\frac{dy}{dx} = -\frac{y}{1 + 2y + x + 3e^{3y}\cos(e^{3y})}$$

Title	Parametric Differentiation
Author	AprilDolphin
Date	11/11/2024

Definition of Parametric Functions
A parametric function is a function that express relationships of 2 variables using a third variable, in which the third variable is called a parameter. Example of a Parametric Function as follows $y = t^2 + 5t - 4$ $x = 2t + 6$ In the above case, the relationship between x and y is expressed using parameter t.
Simplest examples of applications of parametric functions: Kinematics Modeling – where a flying object (such as a bird or fighter jet) trajectory creates a situation where the object's forward motion speeds up and the height follows a curve that goes up and down repeatedly. In this above case, it would be extremely difficult to express the variables in the non-parametric way and using a parametric function for kinematic modelling is appropriate in this case in the following manner. The forward motion (being in acceleration) over time can be written as $x(t) = 0.8t + 3t^2$ The altitude of the flying object in up and down motion over time can be written as $y(t) = 5 + \cos(t)$

Differentiation involving parametric functions can be done using a technique derived from chain rule.

Let x be the first variable, y be the second variable and t be the parameter involved and our goal is to find the change in y with respect to x.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dt} \times \frac{dt}{dx}$$

1. Find $\frac{dy}{dx}$ given that $y = 4t^2 + 3t - 1$ and $x = 3t^4 - 2t$

$$\frac{dy}{dt} = 8t + 3$$

$$\frac{dx}{dt} - 12t^3 - 2$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{(8t + 3)}{(12t^3 - 2)}$$

2. Find $\frac{dy}{dx}$ for the parametric equation $x = 2\sin(t)$ and $y = 3\cos(t)$

$$\frac{dy}{dt} = -3\sin(t)$$

$$\frac{dx}{dt} = 2\cos(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\sin(t)}{2\cos(t)} = -\frac{3}{2}\tan(t)$$

3. A curve is defined parametrically by $x = \frac{2t}{t+1}$ and $y = \frac{t^2}{t+1}$.
   Find $\frac{dy}{dx}$.

Using parametric differentiation, we can find the derivatives of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ as follows.

Quotient rule where $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

$$\frac{dy}{dt} = \frac{(t+1)(2t) - 1(t^2)}{(t+1)^2} = \frac{2t^2 + 2t - t^2}{(t+1)^2} = \frac{t^2 + 2t}{(t+1)^2}$$

$$\frac{dx}{dt} = \frac{(t+1)(2) - 2t(1)}{(t+1)^2} = \frac{2t + 2 - 2t}{(t+1)^2} = \frac{2}{(t+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 + 2t}{(t+1)^2} \times \frac{(t+1)^2}{2} = \frac{t^2 + 2t}{2}$$

Title	Integration by Substitution
Author	AprilDolphin
Date	2/10/2024

Basic Introduction to Integration by Substitution
Integration by Substitution corresponds to chain rule as taught in differentiation and as a result it is sometimes also called the reverse chain rule as well.

Important Concepts to Note as I would be using the concept quite frequently here:

$$1 \div \frac{dy}{dx} = \frac{dx}{dy}$$

Instructions to Integration by Substitution.

Step 1. Examine the questions carefully and define the value of u as mentioned in the question.

Step 2. Differentiate u with respect to x.

Step 3. Express dx in terms of du

Step 4. Rewrite the original integral in terms of u and du

Step 5. Integrate according to relevant formula

Step 6. Replace u with the original expression of x in step 1.

Example 1.
Evaluate $\int x^2\,(x^3+4)^8\,dx$

Step 1. Set $u = (x^3+4)$

Step 2. Differentiate u with respect to x: $2x^2$

Step 3. Express dx in terms of du: $\frac{du}{dx} = 3x^2$, hence, $\frac{dx}{du} = \frac{1}{3x^2}$

Step 4 Rewrite the original integral in terms of u and du: $\int x^2\ u^8 \frac{dx}{du}\,du = \int x^2\ u^8 \left(\frac{1}{3x^2}\right)\,du$

Step 5: Integrate according to relevant formula: $\int \frac{1}{3} u^8 \, du = \frac{1}{3} \times \frac{u^{8+1}}{8+1} = \frac{1}{3}\left(\frac{u^9}{9}\right) = \frac{u^9}{27}$

Step 6: Replace u with the original expression of x in step 1: $\frac{u^9}{27} = \frac{(x^3+4)^9}{27} + C$

Question 1.
Evaluate $\int 2x(x^2+2)^4 \, dx$

Set $u = x^2 + 2$

Hence, $\frac{du}{dx} = 2x$

$$\frac{dx}{du} = \frac{1}{2x}$$

$\int 2x \ (u)^4 \frac{dx}{du} \, du = \int 2x \ u^4 \frac{1}{2x} \, du$

$$\int u^4 \ du = \frac{u^{4+1}}{4+1} = \frac{u^5}{5} = \frac{(x^2+2)^{4+1}}{4+1} = \frac{(x^2+2)^5}{5} + C$$

Question 2.
Evaluate $\int (3x-2)^4 \, dx$

Set $u = 3x - 2$

$\frac{du}{dx} = 3$, therefore $\frac{dx}{du} = \frac{1}{3}$

$$\int \frac{1}{3} u^4 \ du$$

$$\frac{1}{3}\left(\frac{u^5}{5}\right) = \frac{(3x^2-2)^5}{15} + C$$

Question 3.

Evaluate $\int \frac{x-2}{(x+4)^2}\,dx$ using the substitution $u = x + 4$

$$\int \frac{(x+4)-6}{(x+4)^2}\,dx$$

Set $u = x + 4$

$$\int \frac{u-6}{u^2}\,du =$$

$$\int \frac{u}{u^2} - \frac{6}{u^2}\,du =$$

$$\int \frac{1}{u} - \frac{6}{u^2}\,du =$$

$$\ln|u| - \int 6u^{-2}\,du =$$

$$\ln|u| - \frac{6u^{-2+1}}{-2+1} =$$

$$\ln|u| - \frac{6u^{-1}}{-1} = \ln|u| + \frac{6}{u} = \ln|x+4| + \frac{6}{x+4} + C$$

Title	Integration by Parts
Author	AprilDolphin
Date	26/10/2024

Formula for Integration by Parts

$$\int u\,dv = uv - \int v\ du$$

In any integration by parts application, we choose the value of u using a rule where the type of function mentioned first has the highest priority and the one mentioned second has lower priority and so on, the one mentioned the last will have the lowest priority.

The rule is called
"Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, Exponential"

Highest Priority

Lowest Priority

In short, the rule is called the L.I.A.T.E rule.
The purpose of complying to the above rule is to ensure the function we selected is made easy by integration by parts rather than being made even more complicated. As seen with the below example.

Example 1.

$$\int x\,e^x\ dx$$

If we select $u = x$, in compliance with the rule mentioned above, we literally get the following:

$u = x$ and $dv = e^x$

$$v = \int e^x = e^x$$

$\int v = e^x$

$du = 1\,dx$

As a result, we combined them together using the integration by parts formula

$$\int u \, dv = uv - \int v \ du$$

$$x\, e^x - e^x(1) = xe^x - e^x + C$$

However, if we reversed the roles of u and dv above, we get the following

$u = e^x$, $dv = x$

$du = e^x$, $v = \frac{x^2}{2}$

Combining them together, using the integration by parts formula, we get the following integral, literally,

$$e^x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \ e^x \, dx$$

Which becomes as difficult or even harder to integrate compared to the function we first being with and that's not what we want.

Question 1.
Find $\int \ln x \ dx$

Which can be rewritten as

$$\int \ln x\,(1)\, dx$$

Using the L.I.A.T.E rule explained earlier, we can choose $u = \ln x$ and therefore, $dv = 1$.

$u = \ln x$
$v = x$

$$\int x\frac{1}{x}\, dx \ = \int 1 \ dx = x$$

Combining them, we get $x\ln(x) - x + C$
Question 2.
Find $\int x^2 \ln x \ dx$

Selecting $u = \ln x$ and therefore $dv = x^2$

$$v = \int x^2 = \frac{x^3}{3}$$

$$\int v\ du = \int \frac{x^3}{3}\left(\frac{1}{x}\right) = \int \frac{1}{3}x^2 = \frac{\left(\frac{1}{3}\right)}{2+1}x^{2+1} = \frac{1}{9}x^3$$

$$\ln(x)\left(\frac{x^3}{3}\right) - \frac{x^3}{9} + c$$

Question 3.
Find $\int x \cos x\ dx$

Selecting $u = x$ and therefore $dv = \cos x$

$$v = \int \cos x = \sin x$$

$$\int v\ du = \int \sin(x)\ 1\ dx = -\cos x$$

$$x \sin x + \cos x + C$$

Question 4.
Find $\int \tan^{-1} x\ \ dx$

Select $u = \tan^{-1} x$ and $dv = 1\ dx$

$$v = x$$

$$\int v\ du = \int x\left(\frac{1}{x^2+1}\right) = \int \frac{x}{x^2+1} = 0.5\ln(x^2+1)$$

$$x\tan^{-1}(x) - 0.5\ln(x^2+1)$$

Title	Differential Equations – Basic Concepts and First Order Differential Equations (Direct Integration and Separation of Variables)
Author	AprilDolphin
Date	30/10/2024

Differential Equations are equations that contain derivatives, where the so-called "Solutions" are the function(s) that can satisfy the derivatives. The act of finding the indefinite integral of derivatives is generally considered solving simple differential equations.

Concepts to understand in any differential equations

- Order – the order of a differential equation is simply the order of the highest derivative in the differential equation.
- Degree – the degree of a differential equation is simply the power of the highest derivative of the differential equation.

Example

1. Given the following three examples, state the order and degree of the differential equation:

   (a) $x\left(\frac{d^3y}{dx^3}\right) + (3y)^2 = 6x - 8$

   (b) $3x(y'')^2 - y^2y' = 0$

   (c)$x\left(\frac{dy}{dx}\right) + 8\left(\frac{d^2y}{dx^2}\right)^3 = 3x - 5y$

1(a) Highest derivative is the third derivative – third order differential equation
Highest power of the highest derivative is 1 – first degree differential equation
∴ Third order, first degree differential equation.

1(b) Highest derivative is the second derivative – second order differential equation
Highest power of the highest derivative is 2 – second degree differential equation
∴ Second order, second degree differential equation

1(c) Highest derivative is the second derivative – second order differential equation
Highest power of the highest derivative is 3 – third degree differential equation
∴ Second order, third degree differential equation.

After understanding the above, we can go on and understand the concept of independent and dependent variables.

Independent variables in any differential equation are the variables with respect to where differentiation occur and dependent variables are the variables being differentiated.

In the below example, identify the dependent and independent variables.
Example

2.

y is the dependent variable

$$x\left(\frac{d^3y}{dx^3}\right) + (3y)^2 = 6x - 8$$

x is the independent variable

Q1. Solve the following differential equation using direct integration

(a) $\frac{\mathrm{dy}}{\mathrm{dx}} = \ln x$

Move dx to the other side by multiplying both sides by dx to get the following
$dy = \ln x \, dx$

Integrate both sides to get

$$\int dy = \int \ln x \;\, dx$$

Using integration by parts for the right-hand side, we get the following

Set $u = \ln x$, $dv = 1$
$v = x$

$$du = \frac{1}{x}$$

$$\int v \;\, du = \int \frac{x}{x} = \int 1 = x$$

$$y = x(\ln x\,) - x + C$$

Q2. Solve the following differential equation by separation of variables.

$$x + y\left(\frac{dy}{dx}\right) = 2$$

$$y\left(\frac{dy}{dx}\right) = 2 - x$$

Multiply both sides by dx

$$y\ dy = 2 - x\ dx$$

Integrate both sides to get the following

$$\int y\ dy = \int 2 - x\ dx$$

$$\frac{y^2}{2} = 2x - \frac{x^2}{2}$$

$$y^2 = 4x - x^2 + C$$

Q3. Solve the following differential equation by separation of variables.

$$\frac{dy}{dx} = \frac{x^2}{y^3}$$

Cross-multiply both sides denominator and we get the following

$$y^3\ dy = x^2\ dx$$

$$\int y^3\ dy = \int x^2\ dx$$

$$\frac{y^4}{4} = \frac{x^3}{3}$$

Cross multiply again to get

$$3y^4 = 4x^3$$

Divide both sides by 3

$$y^4 = \frac{4}{3}x^3 + C$$